

Multi-partite entanglement detection with non symmetric probing

Luca Dellantonio, Sumanta Das, Jürgen Appel, and Anders S. Sørensen
*The Niels Bohr Institute, University of Copenhagen,
 Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

(Dated: October 7, 2016)

We show that spin squeezing criteria commonly used for entanglement detection can be erroneous, if the probe is not symmetric. We then derive a lower bound on spin squeezing for separable states in spin systems probed asymmetrically. Using this we further develop a procedure that allows us to verify the degree of entanglement of a quantum state in the spin system. Finally, we apply our method for entanglement verification to existing experimental data, and use it to prove the existence of tri-partite entanglement in a spin squeezed atomic ensemble.

Entanglement is a fundamental resource for proving non-classicality of nature, and is a key feature for developing quantum technologies. It was first used experimentally for proving the break down of local realism [1, 2], and has now become a practical ingredient in quantum information science and metrology [3]. As such, it is crucial for quantum information protocols to discern between separable and entangled systems. Unfortunately, this is not always straightforward; even though several criteria exist [4, 5], it may be hard to experimentally verify that a state is entangled. Hence, there is a need for simple, practical procedures for proving that a system is entangled. For instance, spin squeezing [6–8] is a criterion which has often been used in multi-particle systems [9]. One of its major advantages is that it relies on two observables only: the mean spin and the fluctuations perpendicular to it. This makes it ideally suited for systems where there is not complete control over all parameters of the system. Measuring that the noise is squeezed below a certain bound is then a sufficient criterion for proving entanglement [10–18]. Due to the simplicity of this approach, it has been employed to show even multi-partite entanglement in a wide range of experiments [19–23]. However, an inherent assumption in the entanglement criteria based on spin squeezing is that all particles are probed with equal strength. In many practical situations this is a very good approximation [20, 22], but in others this is far from reality. As an example, consider a Gaussian beam probing a collection of trapped atoms. If the waist of the laser beam is much smaller than the size of the cloud, the atoms will have an inhomogeneous interaction with the light. This asymmetry lead to minor modifications of the interaction dynamics if suitable weighted operators are introduced [24, 25]. For entanglement detection in the ensemble, however, the effect of such asymmetry is quite non-trivial.

In this letter, we consider the effect of asymmetric probing on the entanglement criteria. We first consider a simple generalization of the standard squeezing criterion and show that it is no more an effective method for testing entanglement with asymmetric probing. To overcome this problem we develop new entanglement criteria, that can accommodate the asymmetric probing of the particles. We show that our criteria are sufficient for detection of bi-partite entanglement as well as higher order multi-partite entanglement. Finally, we apply

these criteria to a recent experiment [26] to show existence of tri-partite entanglement in an atomic ensemble of cold, asymmetrically probed Cs atoms.

In spin systems, the collective spin operator $\vec{J} = \sum_i \vec{j}_i$ is typically used to represent the observables of the system. Here, operators \vec{j}_i are the standard (pseudo-)spin operators used to describe the individual two level systems with states $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$, (eigenstates of $j_{z,i}$). In reality, however, the probing may differ significantly from particle to particle, so that one rather measures the *weighted spin operator*

$$\vec{S} = \sum_{i=1}^N \eta_i \vec{j}_i, \quad (1)$$

which accounts for the different probing strengths η_i of the N particles.

To characterize spin squeezing Wineland *et al.* [6] introduced the squeezing parameter ξ^2 , which describes the improvement in spectroscopic resolution compared to using coherent spin states $|\text{CSS}\rangle = \bigotimes_{i=1}^N |\uparrow\rangle_i$. Generalizing the derivation of Ref. [6], we find that in the case of inhomogeneous coupling the parameter ξ^2 takes the form

$$\xi_I^2 = \frac{(\Delta S_x)^2}{\langle S_z \rangle^2} \left[\frac{(\Delta S_x)_{\text{CSS}}^2}{\langle S_z \rangle_{\text{CSS}}^2} \right]^{-1}, \quad (2)$$

where averages marked with “CSS” refer to the state $|\text{CSS}\rangle$, and the subscript “I” stands for inhomogeneous. The standard forms $\xi^2 = [\langle J_z \rangle_{\text{CSS}}^2 / (\Delta J_x)_{\text{CSS}}^2] / [\langle J_z \rangle^2 / (\Delta J_x)^2] = N [(\Delta J_x)^2 / \langle J_z \rangle^2]$ of the squeezing parameter is found by setting all coefficients $\eta_i = 1$ in Eq. (1), so that $\vec{S} \rightarrow \vec{J}$. Under this constraint $\xi^2 < 1$ is a sufficient condition for proving entanglement in the ensemble [10]. As proved in the supplemental material, it is possible to find separable states for which $\xi_I^2 \rightarrow \frac{1}{N}$. This implies that the simple squeezing parameter in Eq. (2) cannot be used for entanglement detection with asymmetric probing.

In the following we develop new entanglement criteria for spin systems probed asymmetrically. For this purpose we use the Lagrange multiplier method of Ref. [11]. We are interested in minimizing the variance of the spin along x for a given value of the mean spin along z . We therefore minimize the Lagrange function $\Gamma(\mu) \equiv (\Delta S_x)^2 - \mu \langle S_z \rangle$, where

μ is a Lagrange multiplier. In this way a condition on entanglement for a separable state can be established by finding a lower bound of $(\Delta S_x)^2$ for a given value of $\langle S_z \rangle$. The expectation values of the operators S_x , S_x^2 , and S_z are evaluated for a separable state density matrix defined by $\rho = \sum_k p_k \rho_1^{(k)} \otimes \dots \otimes \rho_N^{(k)}$, where the p_k 's are positive numbers such that $\sum_k p_k = 1$. In addition, for any k and $i = 1, \dots, N$, $\rho_i^{(k)}$ is the density matrix of the i -th particle. With this ρ , we minimize the Lagrange Γ function and obtain the following entanglement criterion:

$$\langle S_z \rangle = \frac{1}{2} \left(\sum_{\mu \leq \eta_i} \mu + \sum_{\mu > \eta_i} \eta_i \right) \quad (3a)$$

$$(\Delta S_x)^2 \geq \frac{1}{4} \left(\sum_{\mu \leq \eta_i} \mu^2 + \sum_{\mu > \eta_i} \eta_i^2 \right). \quad (3b)$$

The summations in the above two equations are to be taken over $\{\eta_i\}_{i=1}^N$ depending on whether they are smaller or bigger than μ , as indicated explicitly. For details, see the supplemental material. Thus we find that to every μ there is an associated state such that, we get the lowest possible variance $(\Delta S_x)^2$ for that value of $\langle S_z \rangle$. This minimum is obtained by having the weakly probed particles fully aligned along the z -axis, whereas the more strongly coupled particles have their spins rotated away from this direction, such that it points partially along the $\pm x$ -axis. As a special case of the results reported here, we consider $\eta_i = 1$ for all i (or, equivalently, $\mu \leq \min\{\eta_i\}$). Eqs. (3a) and (3b) then gives, $\langle S_z \rangle = \mu N/2$, $(\Delta S_x)^2 = \mu^2 N/4$ and we recover the standard quadratic curve $(\Delta S_x)^2 \geq \langle S_z \rangle^2 / N$ used to define the entanglement criterion $\xi^2 < 1$ with symmetric probing [11].

It is imperative to underline the importance of the coefficients η_i . Even though Eqs. (3a) and (3b) give the entanglement criterion we were aiming for, it strongly depends on the distribution of η_i . This thus, results in two important consequences: firstly, as η_i depends on the experimental setup, we cannot define a single squeezing parameter which applies to all experiments. Secondly, we need to know each of these coefficients or, alternatively, their probability distribution function $p(\eta)$ to evaluate the entanglement criterion. Notice that if the positions of the particles are fluctuating in time, their contribution to the measurement will be proportional to the average of the ensemble realizations. Hence Eqs. (3a) and (3b) should be replaced by their average values, which can be determined from the probability distribution function of the coefficients. This in turn means that, as far as $p(\eta)$ remains unchanged during the measurement, fluctuations do not represent an issue for our entanglement criteria.

For illustrating how Eqs. (3a) and (3b) can be used as an entanglement criterion, we introduce a simple example for $p(\eta)$. Let us consider an atomic ensemble in a cylindrical box with radius L . A Gaussian probing beam centred in the middle then interrogates the atoms along the axis of the box. We assume that the particles are uniformly distributed, so that we have

the probing coefficient $\eta(x) = e^{-\frac{x^2}{\sigma^2}}$, where x is the distance from the axis of the laser. In the interval $\eta \in [e^{-\frac{1}{\nu^2}}, 1]$ we have $p(\eta) = \nu^2/\eta$, where $\nu = \sigma/L$ is the only parameter characterizing such model. Inserting this into Eqs. (3a) and (3b) we find the entanglement criterion, shown by the solid black line in Fig. 1 (A). This curve represents the lowest variance one can attain for a separable state, and thus if a variance lower than this is measured for the ensemble, it signifies that (at least) bi-partite entanglement is present in the system. Note that the mean spin and the variance are normalized to the value obtained for the $|\text{CSS}\rangle$. This corresponds to the procedure typically employed in the experiments [23, 27, 28]. As evident from the plot, the dashed black line which is ob-

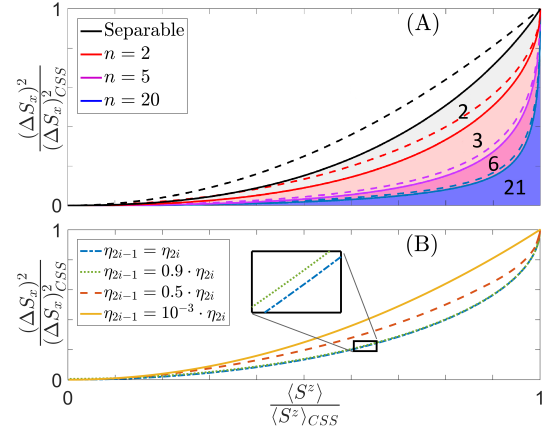


FIG. 1. (A) Entanglement criteria for homogeneous (dashed) and inhomogeneous (solid) probing for an atomic ensemble in a cylinder probed by a narrow Gaussian laser beam ($\nu = 0.3$). The black, red, magenta and blue curves correspond, respectively, to the minimum possible variance when allowing for separable, $n = 2, 5$, and 20 particle-entangled states. The numbers between the lines in the figure therefore express the smallest possible group of entangled atoms in the corresponding regions. (B) Lower bounds for the normalized variance with respect to the average spin for a pair of atoms. The different curves correspond to different values of the coefficients η_{2i-1} and η_{2i} , and were determined using a general wave function $|\psi\rangle = C_{\uparrow\uparrow}|\uparrow\uparrow\rangle + C_{\uparrow\downarrow}|\uparrow\downarrow\rangle + C_{\downarrow\uparrow}|\downarrow\uparrow\rangle + C_{\downarrow\downarrow}|\downarrow\downarrow\rangle$. Along with η_{2i-1} and η_{2i} , $|\psi\rangle$ fully specify the state of the system; we can therefore use the complex numbers C_{kl} ($k, l = \uparrow$ or \downarrow) as variables in the Lagrange Γ function, and obtain the minimal noise for each value of the mean spin, and η_1 and η_2 . Letting the fraction η_{2i-1}/η_{2i} vary continuously in the interval $[0, 1]$, we find that the minimal noise is obtained for $\eta_1 = \eta_2$.

tained for homogeneous probing of the atomic ensemble significantly differs from the present case. This clearly shows the inadequacy of the squeezing parameter ξ^2 as an entanglement criterion in the case of inhomogeneous probing.

Above we have established a criterion for showing the existence of entanglement. We now extend this to multi-partite entanglement by allowing the particles to be entangled in groups of n atoms. Thus, the violation of the minimum obtainable variance for n -particles will identify samples with at least $(n + 1)$ entangled atoms. Assuming for simplicity that the

total number of particles fulfil $N = ln$ for some integer l , we define the density matrix containing at most n entangled particle to be given by

$$\rho = \sum_k p_k \bigotimes_{i=1}^{N/n} \rho_{[n(i-1)+1], \dots, ni}^{(k)} \quad (4)$$

As before, the p_k 's represent some probabilities, and $\rho_{[n(i-1)+1], \dots, ni}^{(k)}$ is the density matrix of n entangled particles for any $k, i = 1, \dots, N/n$. In general, we should allow for permutations between the atoms, but this does not change the result [29]. Moreover, generalizations to non integer N/n can be done, as in the Ref. [18]. As before we evaluate the average spin $\langle S_z \rangle$ and the variance $(\Delta S_x)^2$ for the density matrix in Eq. (4). We then minimize the Lagrange function to find the $(n+1)$ particle entanglement criterion.

We start by considering the simplest case of $n = 2$ entangled atoms, for which $\rho = \bigotimes_{i=1}^{N/2} \rho_{2i-1, 2i}$. Generalizations to any n can be done with minor adjustments which we specify later. For clarity, we have omitted the summation over the index k , that would in the end disappear from the equivalent of Eqs. (7a) and (7b). Using this density matrix, it is possible to obtain $\langle S_z \rangle = \sum_i \langle S_z \rangle_{2i-1, 2i}$ and $(\Delta S_x)^2 \geq \sum_i (\Delta S_x)_{2i-1, 2i}^2$. Therefore a first expression for the Γ function is given by

$$\Gamma = \sum_{i=1}^{N/2} \left\{ \left[\frac{\eta_{2i-1}^2 + \eta_{2i}^2}{4} \right] \frac{(\Delta S_x)_{2i-1, 2i}^2}{(\Delta S_x)_{\text{CSS}; 2i-1, 2i}^2} - \mu \left[\frac{\eta_{2i-1} + \eta_{2i}}{2} \right] \frac{\langle S_z \rangle_{2i-1, 2i}}{\langle S_z \rangle_{\text{CSS}; 2i-1, 2i}} \right\}, \quad (5)$$

where we have written it by renormalizing with respect to the state $|\text{CSS}\rangle$, using $(\Delta S_x)_{\text{CSS}; 2i-1, 2i}^2 = (\eta_{2i-1}^2 + \eta_{2i}^2)/4$ and $\langle S_z \rangle_{\text{CSS}; 2i-1, 2i} = (\eta_{2i-1} + \eta_{2i})/2$. In Eq. (5), all contributions in the sum refer to two (possibly entangled) particles only. This means that by minimizing all of them independently, we obtain a global limit for the whole state. Let us therefore consider the contribution of a single pair described by the coefficients η_{2i-1} and η_{2i} . The minimum that the normalized variance $(\Delta S_x)_{2i-1, 2i}^2 / (\Delta S_x)_{\text{CSS}; 2i-1, 2i}^2$ can achieve for a given value of the mean spin $\langle S_z \rangle_{2i-1, 2i} / \langle S_z \rangle_{\text{CSS}; 2i-1, 2i}$ depends on the ratio η_{2i-1} / η_{2i} . By minimizing the noise numerically we find this minimum, shown in Fig. 1 (B). As seen in the plot, the minimum is achieved when the coefficients are

equal: $\eta_{2i-1} = \eta_{2i}$. This in turn means that the corresponding minimization curve is already known [11]

$$\frac{(\Delta S_x)_{2i-1, 2i}^2}{(\Delta S_x)_{\text{CSS}; 2i-1, 2i}^2} = 1 - \sqrt{1 - \left(\frac{\langle S_z \rangle_{2i-1, 2i}}{\langle S_z \rangle_{\text{CSS}; 2i-1, 2i}} \right)^2}. \quad (6)$$

Inserting Eq. (6) into Eq. (5), and using that the coefficients η_i are pairwise equal, we can rewrite the Lagrange Γ function for pairs of entangled atoms and minimize it for a given μ . In this way we find that

$$\frac{\langle S_z \rangle}{\langle S_z \rangle_{\text{CSS}}} = \left(\sum_{i=1}^{N/2} \eta_i \right)^{-1} \sum_{j=1}^{N/2} \mu \frac{2\eta_j}{\sqrt{\eta_j^2 + 4\mu^2}} \quad (7a)$$

$$\frac{(\Delta S_x)^2}{(\Delta S_x)_{\text{CSS}}^2} = \left(\sum_{i=1}^{N/2} \eta_i^2 \right)^{-1} \sum_{j=1}^{N/2} \eta_j^2 \left[1 - \frac{\eta_j}{\sqrt{\eta_j^2 + 4\mu^2}} \right] \quad (7b)$$

identifies a lower bound on $(\Delta S_x)^2 / (\Delta S_x)_{\text{CSS}}^2$ for the given $\langle S_z \rangle / \langle S_z \rangle_{\text{CSS}}$, assuming that the particles are only entangled in pairs (for details, see the supplemental material). Hence the violation of this bound is a sufficient criterion for proving tripartite entanglement. For the model introduced above, we plot this criterion as the red solid line in Fig. 1 (A). Measurement results below this line will therefore signify samples with at least three entangled particles. The red dashed curve corresponds to the case of symmetric probing of the ensemble. As for the separable case, the difference between them is evident.

For higher multi-partite entanglement ($n > 2$), the procedure for obtaining the criteria is the same. The only differences reside in finding the condition on η_i for minimizing the Lagrange function, and the analogue of Eq. (6) for $n > 2$. To do this, we need to determine the combination of coefficients $\{\eta_1, \dots, \eta_n\}$ which minimize the Lagrange function for n particles. For small n this can be done numerically by minimizing the Lagrange function. The complexity of the minimization, however, grows exponentially with n making the problem intractable for large values. We have explicitly done the minimization for $n = 3$, and we find that the minimum is obtained for the coefficients η_1, η_2 , and η_3 being equal, similar to the $n = 2$ case. Assuming this to be a general property valid for all n , we can seek the equivalent of Eq. (6). For $n > 2$ a tight lower bound on the variance with respect to the average spin is not known. However, since we consider all η_i to be the same, we can use the results of Ref. [11], which give two possible methods. The first one is to exploit the lower bound:

$$\frac{(\Delta S_x)^2}{(\Delta S_x)_{\text{CSS}}^2} \geq 1 + \frac{n}{2} \left[1 - \frac{\langle S_z \rangle^2}{\langle S_z \rangle_{\text{CSS}}^2} \right] - \sqrt{\left[1 + \frac{n}{2} \left(1 - \frac{\langle S_z \rangle^2}{\langle S_z \rangle_{\text{CSS}}^2} \right) \right]^2 - \frac{\langle S_z \rangle^2}{\langle S_z \rangle_{\text{CSS}}^2}}. \quad (8)$$

This condition is not tight since in general one cannot find

a state which reaches the bound. For large n , however, the

bound in Eq. (8) is close to the true minimum. An alternative method is to use a numerical procedure, which is efficient for any even n and represents a tight lower bound [11]. With these methods it is possible to rewrite the equivalent of Eq. (5) for n entangled particles, as a function made of $l = \frac{N}{n}$ n -uplets contributions. These, in a similar way to the $n = 2$ case, can be minimized one by one in order to find all the other multi-partite entanglement criteria, i.e. the equivalent of equations (7a) and (7b) for $n > 2$. Examples of these criteria are shown in Fig.1 (A) for the simple model described above. The solid curves correspond to the lower bounds for the variance that states with $n = 5$ (magenta) and $n = 20$ (blue) entangled atoms cannot go below. The first one is obtained through Eq. (8), the latter using the mentioned numerical procedure.

As an application we revisit the experimental results of Ref. [26], and utilize our criteria to verify entanglement of the atoms. In the experiment, an ensemble of $\gtrsim 10^5$ Cs atoms was cooled and trapped in a Far Off Resonant Trap (FORT). These atoms were then probed with a two colour quantum non demolition scheme [25, 30–33], that allows measuring the variance and average spin. For applying our entanglement criteria we need to know the atomic distribution in the ensemble and the resulting statistics $p(\eta)$ of the coefficients η_i . We assume a thermal distribution of the atoms in the ensemble of the form $\rho(r, z) \propto re^{-\frac{V(r, z)}{k_B T}}$ [34]. Here, T is the temperature, r represents the distance from the beam's axis, k_B is the Boltzmann constant and z is the longitudinal position. The potential $V(r, z)$ generated by the Gaussian FORT beam is of the form $V(r, z) = V_0 \frac{\omega_t^2}{\Omega_t(z)^2} \exp\left[-\frac{2r^2}{\Omega_t(z)^2}\right]$. In this equation, V_0 is the minimum of the potential, ω_t the waist of the trapping beam, and $\Omega_t(z)$ the spatially varying spot size. In the experiment the sample is well described by a thermal distribution in the radial direction r only, not in the longitudinal one z , since the atoms are loaded into the FORT at a well defined position and do not have time to expand to the full length of the system. Thereby, the entanglement criteria we derive with a thermal distribution are lower than the real ones, making their violation harder. To find the coefficients η_i we use that the intensity of the probe in the experiment was described by a Gaussian profile [26]

$$\eta(r, z) = \frac{\omega_p^2}{\Omega_p(z)^2} e^{-\frac{2r^2}{\Omega_p(z)^2}}. \quad (9)$$

The parameters ω_p and $\Omega_p(z)$ are the waist and spot size of the probing beam. We can now evaluate the coefficients η_i [29] according to their probability distribution, and find the multi-partite entanglement criteria. This is shown in Fig. 2, where we also include the experimental results of Ref. [26]. All of the required parameters except the temperature can be directly derived from the experiment [26]. To be conservative we chose the value $T = 50 \mu K$, that is slightly higher than typical values in similar setups [34]. This gives a more inhomogeneous distribution of the coefficients η_i and thereby a lower limit. Furthermore, we consider a possible displacement d between the probe and the FORT. Such a misalignment

between the probing and the trapping beams again results in lowering of the curve, making it harder to detect entanglement. In the experiment the overlap between the probe and the FORT was set by optimizing the detected signal from the atoms. We therefore assign an upper limit to the displacement of $d = 11 \mu m$, set by the requirement that the signal is at least 90% of the maximal one. As seen in Fig. 2 and highlighted

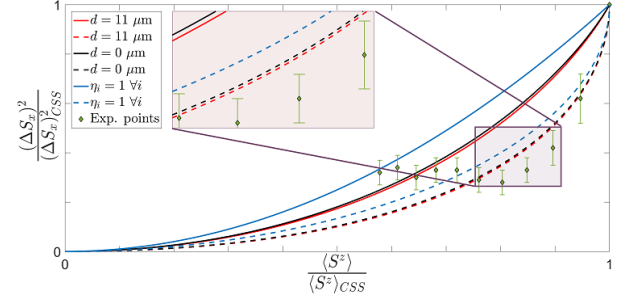


FIG. 2. Entanglement and 3-particle entanglement criteria for the experiment described in Ref. [26]. The solid curves give bounds on the variance assuming a separable state, while the dashed curves give the bound when allowing for bi-partite entanglement. The colours blue (always on top), black and red (always at bottom) correspond respectively to symmetric probing, asymmetric probing with no displacement and asymmetric probing with displacement between the FORT and the probes. Parameters used are: $T = 50 \mu K$, $\omega_t = 50 \mu m$, $\omega_p = 27 \mu m$, and $V_0/k_B = 1.73 \cdot 10^{-4} K$.

in the inset, some of the experimental points lie below the dashed (red & black) curves. Since these curves correspond to the lower limits obtained by allowing bi-partite entanglement, this signifies the presence of tri-partite entanglement among the atoms in the experiment.

In summary, we have considered the problem of verifying entanglement for spin systems subject to asymmetric probing. In this case a naive generalization of the spin squeezing entanglement criteria - derived for homogeneous probing - cannot be used for showing the presence of entanglement in the system. We have explicitly derived new criteria for bi-partite and multi-partite entanglement. A key feature of our procedure is that it can easily be adapted to any setup, once we know the probability distribution function of the coupling coefficients. We have demonstrated this by applying it to the experimental data of Ref. [26], proving tri-partite entanglement among the atoms in the ensemble. Our criteria for characterization based on spin squeezing thus provides effective means of detecting different degrees of entanglement in spin system experiments like atomic ensembles. The procedure is, however, equally applicable to other physical systems [20, 35–37]. Moreover, since the improvement of atomic clocks is related to the spin squeezing parameter, our results suggest that there exist unentangled states which would give a better performance of atomic clocks than $|\text{CSS}\rangle$.

We gratefully acknowledge funding from the European Union Seventh Framework Programme through the ERC Grant QIOS (Grant No. 306576), and the Danish research council. We thank Jörg Müller and Eugene Polzik for fruitful

discussions.

Supplementary Material

VIOLATION OF THE GENERALIZED SQUEEZING INEQUALITY

Here we show that that the generalized squeezing parameter ξ_I^2 can take values strictly smaller than unity for a separable state of N particles. Moreover, we will show that the scaling can be as bad as $\frac{1}{N}$, thus advocating the necessity of finding more suitable entanglement criteria. It is important to recall that, in case of asymmetric probing, the weighted spin operator $\vec{S} = \sum \eta_i \vec{j}_i$ becomes the figure of merit in the measurement of collective variables such as average spin and variance. Firstly, let us consider the density matrix $\rho = \bigotimes_{i=1}^N \rho_i$, that describes a generic separable state without classical correlations. Using this ρ and Jensen lemma [38], it is possible to show that

$$(\Delta S_x)^2 \geq \sum_{i=1}^N \eta_i^2 \langle j_{i,z} \rangle^2 \quad (\text{S1a})$$

$$\langle S_z \rangle = \sum_{i=1}^N \eta_i \langle j_{i,z} \rangle \quad (\text{S1b})$$

More details on how to derive Eqs. (S1a) and (S1b) are given in section S2. These equations, along with the corresponding ones for the coherent spin state |CSS⟩ ($\langle S_z \rangle_{\text{CSS}} = \sum_i \eta_i/2$ and $(\Delta S_x)_{\text{CSS}}^2 = \sum_i \eta_i^2/4$), allow us to obtain at first, a lower bound for the generalized squeezing parameter:

$$\begin{aligned} \xi_I^2(\{\eta_i\}_{i=1}^N; \{\langle j_{i,z} \rangle\}_{i=1}^N) &= \frac{\langle S_z \rangle_{\text{CSS}}^2 (\Delta S_x)^2}{(\Delta S_x)_{\text{CSS}}^2 \langle S_z \rangle^2} = \\ &= \frac{\left(\sum_{i=1}^N \eta_i \right)^2}{\sum_{i=1}^N \eta_i^2} \frac{\sum_{i=1}^N \eta_i^2 \langle j_{i,z} \rangle^2}{\left(\sum_{i=1}^N \eta_i \langle j_{i,z} \rangle \right)^2}. \end{aligned} \quad (\text{S2})$$

Let us consider the distribution $\eta_1 = 1$, $\eta_i = \epsilon$ for all $i = 2, \dots, N$ and take a quantum state $|\psi\rangle$ such that $\eta_i \langle j_{i,z} \rangle = \epsilon$ for all i . Therefore, using Eq. (S2) we get that, up to zeroth order in ϵ ,

$$\xi_I^2(\{\eta_i\}_{i=1}^N; |\psi\rangle) = \frac{1}{N} + o(\epsilon) \quad (\text{S3})$$

This means that there exists a separable state made of N particles for which ξ_I^2 equals $\frac{1}{N}$ up to an infinitesimal quantity. Furthermore one can show that this is a lower bound on ξ_I^2 .

GENERALIZED ENTANGLEMENT CRITERION FOR ASYMMETRIC PROBING

In the following, we are going to derive the optimal lower bound for the variance $(\Delta S_x)^2$, given some value for the average spin $\langle S_z \rangle$ in the case of a separable state. As explained

in the main text, the procedure is to write down the Lagrange function Γ and minimize it. First, we need to derive expressions for both $\langle S_z \rangle$ and $(\Delta S_x)^2$, in terms of the single particles' spin averages. Using the definition of the weighted spin operator in Eq. (1) and knowing that the density matrix is $\rho = \sum_k p_k \rho_1^{(k)} \otimes \dots \otimes \rho_N^{(k)}$, it is easy to obtain:

$$\langle S_z \rangle = \sum_k p_k \sum_{i=1}^N \eta_i \langle j_{i,z} \rangle_k \quad (\text{S4a})$$

$$(\Delta S_x)^2 \stackrel{\dagger}{\geq} \sum_k p_k \sum_{i=1}^N \eta_i^2 \langle j_{i,z} \rangle_k^2. \quad (\text{S4b})$$

Here, $j_{i,z}$ is the component of the i -th particle's spin along the z direction, and $\langle A \rangle_k = \text{Tr} \{ \rho^{(k)} A \}$ for any operator A . The inequality marked with \dagger has been derived by using $\langle j_{i,x}^2 \rangle_k - \langle j_{i,x} \rangle_k^2 \geq \langle j_{i,z} \rangle_k^2$ for spin $\frac{1}{2}$ particles, and Jensen's lemma [38], which states that for a convex function f , $\sum_i f(x_i) \leq f(\sum_i x_i)$. Notice that it is always possible to find a state such that Eq. (S4b) becomes an equality. In fact, whenever all the products between the coefficient η_i and the single particle's average spins $\langle j_{i,z} \rangle_k$ give the same contributions, Jensen inequality is saturated. This means that the entanglement criterion we report in the letter is optimal, since there always exists a separable state that can reach it. Using Eqs. (S4a) and (S4b) we can write down the Γ function

$$\begin{aligned} \Gamma(\mu) &\equiv (\Delta S_x)^2 - \mu \langle S_z \rangle_k = \\ &= \sum_{k,i=1}^N p_k \left\{ \frac{1}{4} \eta_i^2 \langle j_{i,z} \rangle_k^2 - \frac{\mu}{2} \eta_i \langle j_{i,z} \rangle_k \right\}, \end{aligned} \quad (\text{S5})$$

whose only variables are $\langle j_{i,z} \rangle_k$. Taking the derivatives of Eq. (S5), we find the minimum by setting, for all $k, i = 1, \dots, N$

$$\langle j_{i,z} \rangle_k = 1 \quad \text{if } \mu > \eta_i \quad (\text{S6a})$$

$$\langle j_{i,z} \rangle_k = \frac{\mu}{\eta_i} \quad \text{if } \mu \leq \eta_i. \quad (\text{S6b})$$

Notice that both equations (S6a) and (S6b) are independent of k . For this reason, the summation over p_k in Eq. (S5) disappears as a unit factor. The entanglement criterion of Eqs. (3a) and (3b) in the main text is obtained by imposing the conditions for being in the minimum found above.

THE TRI-PARTITE ENTANGLEMENT CRITERION

In this section we are going to formally prove the tri-partite entanglement criterion, namely Eqs. (7a) and (7b). First, let us define the numbers $\tilde{\eta}_i = \frac{\eta_{2i-1} + \eta_{2i}}{2}$, so that we can take the lower bound of Eq. (5):

$$\Gamma \geq \sum_{i=1}^{\frac{N}{2}} \left\{ \frac{\tilde{\eta}_i^2}{2} \frac{(\Delta S_x)_{2i-1,2i}^2}{(\Delta S_x)_{\text{CSS};2i-1,2i}^2} - \mu \tilde{\eta}_i \frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}} \right\}. \quad (\text{S7})$$

The reason for which the previous equation represents a lower bound, is simply that $\frac{\eta_{2i-1}^2 + \eta_{2i}^2}{4} \geq \frac{\eta_i^2}{2}$, with the inequality saturated if and only if $\eta_{2i-1} = \eta_{2i}$. Recalling here Eq. (6),

$$\frac{(\Delta S_x)_{2i-1,2i}^2}{(\Delta S_x)_{\text{CSS};2i-1,2i}^2} \geq 1 - \sqrt{1 - \left(\frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}} \right)^2}, \quad (\text{S8})$$

we can obtain another lower bound of the Lagrange Γ function by substituting Eq. (S8) into Eq. (S7):

$$\Gamma \geq \sum_{i=1}^{\frac{N}{2}} \left\{ \frac{\tilde{\eta}_i^2}{2} \left[1 - \sqrt{1 - \left(\frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}} \right)^2} \right] + \mu \tilde{\eta}_i \frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}} \right\}. \quad (\text{S9})$$

We can now study the function of Eq. (S9) with respect to the variables $\frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}}$. The only minimum is found by setting, for all $i = 1, \dots, \frac{N}{2}$,

$$\frac{\langle S_z \rangle_{2i-1,2i}}{\langle S_z \rangle_{\text{CSS};2i-1,2i}} = \frac{2\mu}{\sqrt{\tilde{\eta}_i^2 + 4\mu^2}}. \quad (\text{S10})$$

Substituting Eq. (S10) into Eq. (S9) we obtain the tri-partite entanglement criterion presented in Eqs. (7a) and (7b) of the main text.

[1] J. Bell, Speakable and unspeakable in quantum mechanics (1987).
[2] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[3] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge university press, 2010).
[4] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[5] O. Gühne and G. Tóth, Physics Reports **474**, 1 (2009).
[6] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Phys. Rev. A **50**, 67 (1994).
[7] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Phys. Rev. A **46**, R6797 (1992).
[8] M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993).
[9] L. Pezzé, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, arXiv:1609.01609.
[10] A. S. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature **409**, 63 (2001).
[11] A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001).
[12] G. Vitagliano, I. Apellaniz, I. L. Egusquiza, and G. Tóth, Phys. Rev. A **89**, 032307 (2014).

[13] P. Hyllus, L. Pezzé, A. Smerzi, and G. Tóth, Phys. Rev. A **86**, 012337 (2012).
[14] G. Vitagliano, P. Hyllus, I. L. Egusquiza, and G. Tóth, Phys. Rev. Lett. **107**, 240502 (2011).
[15] J. Ma, X. Wang, C. Sun, and F. Nori, Physics Reports **509**, 89 (2011).
[16] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **95**, 120502 (2005).
[17] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. Lett. **99**, 250405 (2007).
[18] P. Hyllus, L. Pezzé, and A. Smerzi, Phys. Rev. Lett. **105**, 120501 (2010).
[19] O. Hosten, N. J. Engelsen, R. Krishnakumar, and M. A. Kasevich, Nature **529**, 505 (2016).
[20] J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature **455**, 1216 (2008).
[21] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature **464**, 1170 (2010).
[22] C. F. Ockeloen, R. Schmied, M. F. Riedel, and P. Treutlein, Phys. Rev. Lett. **111**, 143001 (2013).
[23] K. C. Cox, G. P. Greve, J. M. Weiner, and J. K. Thompson, Phys. Rev. Lett. **116**, 093602 (2016).
[24] J. Hu, W. Chen, Z. Vendeiro, H. Zhang, and V. Vuletić, Phys. Rev. A **92**, 063816 (2015).
[25] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Rev. Mod. Phys. **82**, 1041 (2010).
[26] J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjærgaard, and E. S. Polzik, Proceedings of the National Academy of Sciences **106**, 10960 (2009).
[27] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Phys. Rev. Lett. **83**, 1319 (1999).
[28] J. G. Bohnet, K. C. Cox, M. A. Norcia, J. M. Weiner, Z. Chen, and J. K. Thompson, Nature Photonics **8**, 731 (2014).
[29] L. Dellantonio, Entanglement criteria for spin squeezing (master thesis, Copenhagen university, unpublished, 2015).
[30] A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. **85**, 1594 (2000).
[31] M. Saffman, D. Oblak, J. Appel, and E. S. Polzik, Phys. Rev. A **79**, 023831 (2009).
[32] P. J. Windpassinger, D. Oblak, U. B. Hoff, J. Appel, N. Kjærgaard, and E. S. Polzik, New Journal of Physics **10**, 053032 (2008).
[33] P. J. Windpassinger, D. Oblak, P. G. Petrov, M. Kubasik, M. Saffman, C. L. G. Alzar, J. Appel, J. H. Müller, N. Kjærgaard, and E. S. Polzik, Phys. Rev. Lett. **100**, 103601 (2008).
[34] D. Oblak, Quantum state engineering in cold caesium atoms. An entangled tale of non-destructively induced collapse and squeezing of atomic states (PhD thesis, Copenhagen university, Museum Tusculanum, 2010).
[35] R. Auccaise, A. G. Araujo-Ferreira, R. S. Sarthour, I. S. Oliveira, T. J. Bonagamba, and I. Roditi, Phys. Rev. Lett. **114**, 043604 (2015).
[36] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin, and J. I. Cirac, Phys. Rev. A **86**, 012116 (2012).
[37] M. S. Rudner, L. M. K. Vandersypen, V. Vuletić, and L. S. Levitov, Phys. Rev. Lett. **107**, 206806 (2011).
[38] J. Jensen, Acta Mathematica **30**, 175 (1906).